Recall that a <u>δ-ring</u> is a ring R equipped with a map $\delta : R \to R$ s.t.

$$(\ast) \quad \begin{cases} (1)\ \delta(x+y) = \delta(x) + \delta(y) + \frac{x^p + y^p - (x+y)^p}{p} \\ \qquad = \delta(x) + \delta(y) - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} x^i y^{p-i} \\ (2)\ \delta(xy) = x^p \delta(y) + y^p \delta(x) + p\,\delta(x)\delta(y) \\ (3)\ \delta(0) = \delta(1) = 0 \end{cases}$$

Goal: Define prismatic cohomology ~ deform de Rham cohomology w/ lift of Frobenius — all other structures come from this

Let A be a ring w/ lift of Frob. $\varphi : A \to A$, hence:
$$\varphi(x) = x^p + p\,\delta(x).$$
Now we may ~~forget~~ forget φ, but ~~remet~~ remember δ instead.

If A is p-torsion free, φ pins δ down.

Defn. A <u>δ-ring</u> is a pair $(A, \delta : A \to A)$ satisfying $(\ast)$.

Lemma
- If A is a δ-ring, then $x \mapsto \varphi(x) = x^p + p\delta(x)$ gives you a lift of Frob.
- if A is p-torsion free, then $\{\varphi\} \longleftrightarrow \{\delta\}$ is a 1-1 correspondence

Rmk. In a δ-ring (A, δ), we have $\delta(xy) = \varphi(x)\delta(y) + y^p \delta(x)$.
This is a useful identity.

Lemma: $\exists!$ δ-structure on $\mathbb{Z}$.

pf: $\mathbb{Z}$ is p-torsion free w/ only lift of Frob $= \mathrm{id}_{\mathbb{Z}}$, hence:
$$\delta(n) = \frac{\varphi(n) - n^p}{p} = \frac{n - n^p}{p}.$$

Lemma. If (A, δ) is a δ-ring w/ $p^n = 0$ in A for some $n \in \mathbb{N}$. Then $A = 0$ ring.

pf: $0 = \delta(p^n) = \frac{p^n - p^{np}}{p} = p^{n-1} - p^{np-1} = p^{n-1}$ (since $np - 1 \geq n$)

By induction, we get $1 = 0$.

Example: (1) On $\mathbb{Z}$, $\exists!$ δ-structure as we have seen. This is the initial object in $\{\delta\text{-rings}\}$.

(2) $A = \mathbb{Z}[x]$, $\varphi: A \longrightarrow A$, $x \longmapsto x^p + p \cdot g(x)$ gives a δ-ring.

(3) k = perfect field of char p, $W(k)$ = unique p-adically complete and p-torsion free lifting k. $\varphi \rightsquigarrow \delta$-structure.

(4) If A is a $\mathbb{Z}[\frac{1}{p}]$-algebra and $\varphi: A \to A$ any ring map, we get a δ-structure on A.

(5) $A = \mathbb{Z}[x]/(px, x^p)$

Claim: $\exists!$ δ-structure on A w/ $\delta(x) = 0$.

Rmk: An element x in a δ-ring A has <u>rank 1</u> if $\delta(x) = 0$.

The Category of δ-rings.

Construction: For any ring A, define $W_2(A)$ as follows:

as a set $W_2(A) = A \times A$

$$(x, y) + (z, w) = \left(x + z,\ y + w + \frac{x^p + z^p - (x+z)^p}{p}\right).$$

$$(x, y) \cdot (z, w) = (xz,\ x^p w + z^p y^p + p \cdot yw).$$

Lemma. (A, δ) is a δ-ring, the map

$$A \longrightarrow W_2(A)$$
$$a \longmapsto (a, \delta(a))$$

is a ring homomorphism giving a section to $W_2(A) \to A$. $\quad (x, y) \longmapsto x$

Conversely, any such homomorphism gives rise to a δ-structure.

Rmk. If A is p-torsion free, then $W_2(A)$ is the fiber product of

$$\begin{tikzcd} W_2(A) \arrow[r] \arrow[d] & A \arrow[d, "\pi"] \\ A \arrow[r, "\mathrm{Frob} \circ \pi"] & A/p \end{tikzcd}$$

Lemma. The category of δ-rings has all limits & colimits and they are computed on underlying rings.

pf: For limits, it's easy.

For colimits, $\{A_i\}_{i \in I}$ is a diagram of δ-rings.

get maps $A_i \longrightarrow W_2(A_i)$

Take colim:
$$\begin{tikzcd} \operatorname{colim} A_i \arrow[r] \arrow[dr] & \operatorname{colim} W_2(A_i) \arrow[d] \\ & W_2(\operatorname{colim} A_i) \end{tikzcd}$$

gives δ-structure we want.

Rmk: Lemma $\Rightarrow$ $\{\delta\text{-rings}\} \longrightarrow \{\text{rings}\}$ has both left & right adjoint.

Note: Right adjoint $= W(-)$ (Joyal).

Lemma: The free δ-rings $\mathbb{Z}\{x\}$ on a variable x is given by:

$$\mathbb{Z}\{x\} = \mathbb{Z}[x_0, x_1, x_2, \dots]$$

$$\delta(x_i) = x_{i+1}$$

pf. To check $\mathbb{Z}\{x\}$ is a δ-ring: $\varphi(x_i) = x_i^p + p x_{i+1}$ extends uniquely to a Frob lift on $\mathbb{Z}\{x\}$.

Rmk: $\mathbb{Z}\{x\}$ is p-torsion free!

Example:

$$\begin{tikzcd} \mathbb{Z}\{z\} \arrow[r] \arrow[d] & \mathbb{Z}\{x,y\} \arrow[d] \\ \mathbb{Z} \arrow[r] & \mathbb{Z}\{x,y\}/(x^2+y^3+xy)_\delta \end{tikzcd}$$

$z \longmapsto x^2+y^3+xy$, $z \mapsto 0$.

Lemma: A δ-ring, $S \subseteq A$ multiplicative set s.t. $\varphi(S) \subseteq S$.
$\Rightarrow S^{-1}A$ has a unique δ-structure compatible with that of A.

pf. • Assuming A is p-torsion free, then $\exists!$ $S^{-1}A \xrightarrow{\varphi} S^{-1}A$ compatible w/ φ on A.
A p-torsion free $\Rightarrow S^{-1}A$ p-torsion free, hence we get a unique δ-structure on $S^{-1}A$.

In general, can find a free δ-ring F, a multiplicative set $T \subseteq F$ s.t. $\varphi(T) \subseteq T$ and a surjection $F \twoheadrightarrow A$ taking $T \twoheadrightarrow S$.

get a pushout diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & T^{-1}F \arrow[d] \\ A \arrow[r] & S^{-1}A \end{tikzcd}$$

Resulting δ-structure on $S^{-1}A$ does not depend on the choice of F & T. Because there can have at most one δ-structure on $S^{-1}A$ compatible w/ δ on A.

$$\left(\delta\left(\frac{1}{s}\right) = \frac{-\left(\frac{1}{s}\right)^p \delta(s)}{\varphi(s)} \right).$$

Perfect δ-rings

Defn A δ-ring A is *perfect* if $\varphi: A \to A$ is an isom.

Thm The following Cats are equivalent:

(1) Perfect and p-adically complete δ-rings

(2) Perfect $\mathbb{F}_p$-algebras.

via $R \longmapsto R/p$

$W(S) \longleftarrow S$.

Key Lemma: If A is a p-adically complet δ-ring, and $x \in A$ is a p-torsion, i.e., $px=0$. Then $\varphi(x)=0$.

pf. $0=\delta(px) = \varphi(x)\cdot\delta(p) + p^p\delta(x)$

$\delta(p)$: $p$adic unit

It suffices to show $p^p\delta(x)=0$.

$\Leftarrow p^{p-1}(p\delta(x)) = p^{p-1}(\varphi(x)-x^p)$

$= p^{p-2}(\varphi(px) - (px)\cdot p^{p-1}) = 0$

Distinguished elements.

Notation: A is a comm ring $\rightsquigarrow$ $Rad(A)$ = Jacobson Radical of A.

$\boxed{\text{Assume } p\in Rad(A)}$

Defn. An element d in a δ-ring A is <u>distinguished</u> (or <u>primitive</u>) if $\delta(d)\in A^*$.

Lemma. d distinguished $\Longrightarrow$ $\cancel{\delta(d)}$ $\varphi(d)$ distinguished.

pf. $\varphi\circ\delta = \delta\circ\varphi$.

Example (1) (Crystalline Coh.): $A=\mathbb{Z}_{(p)}$, $d=p$, $\delta(p)=1-p^{p-1}\in A^*$.

In fact, for any δ-ring A, $d=p$ is distinguished.

(2) (q-dR Coh.) $A=\mathbb{Z}_p[[q-1]] \circlearrowleft \varphi(q)=q^p$. $d=[p]_q = \frac{q^p-1}{q-1} = 1+q+q^2+\cdots+q^{p-1}$.

Consider $A \longrightarrow A/(q-1) \underset{q=1}{\cong} \mathbb{Z}_p$ δ-map.

$[p]_q \longmapsto p$ $\quad \delta([p]_q)\longmapsto \delta(p)=1-p^{p-1}$ unit.

(3) (Breuil-Kisin Coh.) $A=\mathbb{Z}_p[[u]]$ $\varphi(u)=u^p$ $d=u-p\in A$ is distinguished.

(4) ($A_{\inf}$ coh.) $A=(p,q-1)$-adic completion of $\mathbb{Z}_p[q, q^{1/p}, q^{1/p^2},\cdots]$.

$d=[p]_q = \frac{q^p-1}{q-1}$.

Lemma $A=\delta$-ring, $d\in A$ distinguished, $p,d\in Rad(A)$

$\Rightarrow$ $d\cdot(\text{unit})$ is distinguished

pf. say $u\in A$ is a unit.

$\delta(du) = d^p\delta(u) + u^p\delta(d) + p\,\delta(u)\cdot\delta(d) \in A^*$.

Lemma (Irreducibility lemma) Given a δ-ring A, elts $f,d\in A$ s.t. d is distinguished and $d=f\cdot g$. Assume $p,f\in Rad(A)$. Then f is distinguished, g is a unit

pf. $\delta(d)=\delta(fg) = f^p\delta(g) + g^p\delta(f) + p\delta(f)\delta(g)$

$\Rightarrow g^p\delta(f)\in A^*$.

Lemma Fix a δ-ring A, an element $d\in A$, s.t. $(p,d)\subseteq Rad(A)$.

Then d is distinguished $\Leftrightarrow$ $p\in(d,\varphi(d))$.

Hence "d is distinguished" only depends on (d).

pf: $(\Rightarrow)$ d is distinguished $\Rightarrow$ $\delta(d)\in A^*$ $\Rightarrow$ $\varphi(d) = d^p + p\overset{\text{unit}}{\delta(d)}$

$\Rightarrow p\in(d,\varphi(d))$.

$(\Leftarrow)$ say $p=ad+b\varphi(d)$.

Goal: $\delta(d)$ is a unit $\Leftrightarrow$ $\delta(d)$ is a unit in $A/(p,d)$

$\Leftrightarrow A/(p,d,\varphi(d))=0$.

Proceed by contradiction. ~~We may localize so that $(p,d,\varphi(d))\subseteq Rad(A)$.~~

If not, $\exists\, m\subseteq A$, s.t. $\delta(d)\in m$. ~~Caution:~~

$A\setminus m$ is φ-stable (b/c $p\in Rad(A)$), so A_m has a unique δ-str...

Now, in A_m, we have $p=ad+b\varphi(d) = ad+b(d^p+p\delta(d))$.

$\Rightarrow p(1-b\delta(d)) = (a+b\cdot d^{p-1})\cdot d$.

$$\left.\begin{array}{l}\overbrace{p(1-b\delta(d))}^{\text{distinguished as } \delta(d)\in M} = c\cdot d \\ \text{Irred. Lemma}\end{array}\right\} \Rightarrow d \text{ is distinguished} \Rightarrow \delta(d)\notin M \quad \text{Contradiction}.$$

Upshot: $p\in(d,\varphi(d)) \Leftrightarrow d$ is distinguished.

Lemma ~~R is a perfect $\mathbb{F}_p$-alg.~~

(1) An element $d\in W(R)$ is distinguished iff $a_1\in R$ is a unit where $d=\sum_{i\geq 0}[a_i]p^i$.

(2) Any distinguished element of $W(R)$ is a nonzerodivisor.

(3) If $d\in W(R)$ is distinguished, then $W(R)/(d)$ has bounded p^∞-torsions. In fact $W(R)/(d)[p^\infty] = W(R)/(d)[p]$.

pf. (1): $d=[a_0]+[a_1]p+p^2x$, $\varphi(d)=[a_0^p]+[a_1^p]p+p^2\varphi(x)$.

$$\delta(d)=\frac{\varphi(d)-d^p}{p}\equiv a_1^p \pmod p.$$

(2) Say $d=\sum[a_i]p^i$, $X=\sum_{j\geq 0}[x_j]p^j$ with $\begin{cases} d\cdot x=0 \\ x_0\neq 0\end{cases}$.

Then $\begin{cases} a_0x_0=0 \\ a_1x_0+a_0x_1=0\end{cases} \Rightarrow a_1x_0^2=0 \Rightarrow x_0^2=0 \Rightarrow x_0^p=0 \Rightarrow x_0=0$ b/c perfectness of R.

Contradiction.

(3). We will show: if $p^2x=dy$, then $p\mid y$.

Say $d=\sum[a_i]p^i$, $x=\sum[x_i]p^i$, $y=\sum[y_i]p^i$.

Then modulo p^2, we get $\begin{cases} a_0y_0=0. \\ a_1y_0+a_0y_1=0\end{cases} \Rightarrow y_0=0$ by same argument.

Therefore $p\mid y$ ~~and~~ ~~$p\nmid$~~.

Rmk: When $p\in \mathrm{Rad}(A)$, inverting x necessarily inverts $\varphi^n(x)$ for all n. Hence localization will retain δ-ring str. automatically.

Digression: Derived completions.

Defn $A=$ comm. ring and $f_1,\cdots,f_r\in A$, $I=(f_1,\cdots,f_r)$. An A-complex $M\in D(A)$ is derived I-complete if $\forall f\in I$

$$T(M;f)=R\lim(\cdots\xrightarrow{f}M\xrightarrow{f}M\xrightarrow{f}M) \text{ is } 0.$$

$$\Leftrightarrow M\simeq R\lim_n\left(M\overset{L}{\underset{\mathbb{Z}[x_1,\cdots,x_r]}{\otimes}}\frac{\mathbb{Z}[x_1,\cdots,x_r]}{(x_1^n,\cdots,x_r^n)}\right)=:\widehat{M}.$$

where x_i acts by f_i.

Properties

(1) $\left\{\begin{array}{l}\text{all derived} \\ I\text{-cplt } M\end{array}\right\}\subseteq D(A)$ is a triangulated subcat, closed under product and has a left adjoint $M\longmapsto\widehat{M}$.

(2) M is derived I-cplt $\Leftrightarrow$ each $H^i(M)$ is so.

(3) $\left\{\begin{array}{l}\text{all derived} \\ I\text{-cplt } A\text{-mod.}\end{array}\right\}$ is an abelian subcat in $\{\text{all } A\text{-mod.}\}$

(4) derived Nakayama: if $M\in D(A)$ is derived I-complete. Then $M=0 \Leftrightarrow M\overset{L}{\underset{A}{\otimes}}A/I=0$.

Recall: TFAE: (a) $p \in (I, \varphi(I))$

(b) I is locally gen. by a distinguished elt

(c) $p \in (I^p, \varphi(I))$.

Defn. • δ-pairs $= \{(A, I) \mid A$ δ-ring, $I \subseteq A$ ideal$\}$.

• A prism is a δ-pair (A, I) s.t.

(1) I is locally gen. by a non-zero divisor (defines a Cartier div. on Spec(A))

(2) A is derived-(p, I)-complete.

(3) $p \in (I, \varphi(I))$.

• A map $(A, I) \to (B, J)$ of prisms is (faithfully) flat if

$A/(p,I) \longrightarrow \cancel{B/(p,J)}\ B \overset{L}{\underset{A}{\otimes}} A/(p, I)$ is

(faithfully) flat $\Longleftrightarrow$ supp. in degree 0 and (faithfully) flat.

• A prism (A, I) is called

(1) perfect if A is perfect

(2) crystalline if $I = (p)$

(3) bounded if A/I has bounded p^∞-torsion.

Ex. (1) Any p-torsion free & p-adically cplt δ-ring A gives a prism $(A, (p))$.

(2) all examples from last time.

(3) perfect prisms = perfectoid rings.

Lemma: (A, I) prism. Then $\varphi(I) \cdot A$ is principal and any generator is a distinguished elt.

exercise ⟶ ~~$I \in \mathrm{Pic}(A)$ is a p-torsion elt.~~

pf. write $p = a + b$, $a \in I^p$, $b \in \varphi(I)$.

Claim: b is a generator of $\varphi(I)$. (use irreducibility Lemma).

Lemma. ~~$A/I \to B/J$~~ $(A, I) \to (B, J)$ map of prisms.

$\Rightarrow$ $I \underset{A}{\otimes} B \longrightarrow J$ is an isom.

pf. use irreducibility Lemma.

Perfect Prisms:

Defn. A commutative ring R is perfectoid if $R \cong A/I$ for a perfect prism (A, I).

Ex. (1) A = perfect & p-adically complete δ-ring.

$I = (p)$. $\rightsquigarrow$ $A \cong W(R)$, R perfect $\mathbb{F}_p$-alg. $A/I \cong R$.

(2) $A = \mathbb{Z}_p[x^{1/p^\infty}]^\wedge_{(p,x)}$, $I = (x - p)$. $p \in (x-p, x^p - p)$.

$\varphi(x^{1/p^n}) = x^{1/p^{n-1}}$.

$\rightsquigarrow$ $R = \mathbb{Z}_p[p^{1/p^\infty}]^\wedge_{(p)}$.

Lemma: R perfect $\mathbb{F}_p$-alg., $f \in R$. Then $R[f^\infty] = R[f^{1/p^n}]$ $\forall n$. ("almost zero")

pf. Say $x \in R$, s.t. $f^m \cdot x = 0$,

$\Rightarrow f^m \cdot x^{p^n} = 0$ $\forall n \geq 0$ $\Rightarrow f^{m/p^n} \cdot x = 0 \Rightarrow x \in R[f^{1/p^n}]$

Cor. (A, I) perfect prism. $\Rightarrow$ $A/I[p^\infty] = A/I[p]$

$\Rightarrow$ perfect prisms are bounded.

Thm. The functor $(A, I) \longmapsto A/I = R$ gives an equivalence between $\left\{\begin{matrix}\text{perfect}\\ \text{prisms}\end{matrix}\right\} \xrightarrow{\cong} \left\{\begin{matrix}\text{perfectoid}\\ \text{rings}\end{matrix}\right\}$.

pf. How to recover (A, I) from $R = A/I$.

~~Choose $d \in I$ a generator~~

Claim: $A \cong W(R^\flat)$.

Write $I=(d)$ for a dist. elt d

$\Rightarrow R = A/(d) \Rightarrow R/p = A/(p,d)$.

As A/p is a perfect $\mathbb{F}_p$-alg. we have $\cancel{A/(p,d)}$

$$\begin{array}{ccccc} A/(p,d^{p^2}) & \to & A/(p,d^p) & \xrightarrow{can} & A/(p,d) \\ \downarrow\wr & & \downarrow\wr & & \| \\ \cdots \to R/p & \xrightarrow{\varphi} & R/p & \xrightarrow{\varphi} & R/p \end{array}$$

Taking inverse limit: $(A/p)^{\wedge}_{(d)} \xrightarrow{\sim} R^b$

b/c A/p is d-complete. $\| $ $A/p \Rightarrow A \cong W(R^b)$.

Defn. R perfectoid ring,

(1) The tilt R^b of R is $\varprojlim_{\varphi} R/p$.

(2) $A_{inf}(R) = W(R^b) \circlearrowright \varphi$.

and the map $\theta_R : A_{inf}(R) \longrightarrow R$.

(3) The special fiber $\bar{R} := \varinjlim_{\varphi} R/p$.

Ex. $R = \mathbb{Z}_p\langle p^{1/p^\infty}\rangle$ $A_{inf} = \mathbb{Z}_p[x^{1/p^\infty}]^{\wedge}_{(p,x)}$. $\bar{R} = \mathbb{F}_p$.

Lemma Say R is a perfectoid ring. Then

(1) The Frobenius $R/p \longrightarrow R/p$ is surj.

(2) $\exists\ \pi \in R$ s.t. $\pi^p = p\cdot(\text{unit})$. and $(\pi) = \ker(R/p \to R/p)$.

(3) $\sqrt{pR}$ is a flat ideal and $(\sqrt{pR})^2 = \sqrt{pR}$.

(4) $R[p^\infty] = R[p] = R[\sqrt{pR}]$.

pf. Write $R = A/(d)$, d distinguished, $A = W(R^b)$.

(1) R/p is ~~qu~~ a quotient of A/p.

(2) $d = [a_0] + [a_1]p + [a_2]p^2 + \cdots$ where a_1 is a unit in R^b.

$\Rightarrow d = [a_0] - p\cdot u$ for a unit $u \in A$

$\Rightarrow$ in $R = A/d$, have $[a_0] = p\cdot u$.

Take $\pi = [a_0^{1/p}]$.

(3) Claim: $([a_0^{1/p^n}]_{n\geq 1}) = \sqrt{pR}$ in R.

suffices to show $R/([a_0^{1/p^n}])$ is reduced (hence perfect).

General fact: If S perfect $\mathbb{F}_p$-alg. $I = (f_1, \cdots, f_r) \subseteq S$ ideal,

$\Rightarrow \sqrt{I} = (f_1^{1/p^\infty}, \cdots, f_r^{1/p^\infty})$.

For the rest of the pf, look in the notes.

Properties of perfectoids.

(1) $$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \\ T & & \end{array}$$ diagram of perfectoids $\Rightarrow \widehat{S \otimes^L_R T}$ is perfectoid.

(2) R perfectoid $\Rightarrow S = R/R[\sqrt{pR}]$ is ~~perfe~~ perfectoid & p-torsion free and

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ \bar{R} & \longrightarrow & \bar{S} \end{array}$$ is a fiber square.

(3) R is reduced.

The prismatic site.

Setup: (A, I) = "base prism"

Assume: · $I = (d)$, d dist. elt
· $A/I[p^\infty] = A/I[p^N]$ for some $N >> 0$.

Example:
(1) $A = W(\underset{\text{perfect}}{R})$, $I = (p)$.
(2) $A = \mathbb{Z}_p[[u]] \circlearrowleft \varphi$, $\varphi(u) = u^p$. $I = (E(u))$, $E(u)$ = Eisenstein poly. (e.g. $u^p - p$).
(3) R = pftd ring $\rightsquigarrow (A, I) = (A_{inf}(R), \ker(\theta_R))$.
(4) $A = \mathbb{Z}_p[[q-1]] \circlearrowleft \varphi(q) = q^p$. $I = ([p]_q)$, $[p]_q = \frac{q^p - 1}{q-1} = 1 + q + \cdots + q^{p-1}$

R = formally smooth (A/I)-alg. (ex. p-adically cplt of smooth A/I-alg.).

Defn. The prismatic site of R rel. to A consists of prisms (B, IB) over A + a map $R \to B/IB$ over A/I.

Pictorially:

$$\begin{array}{ccccc} A & \longrightarrow & & & B \\ \downarrow & & & & \downarrow \\ A/I & \longrightarrow & R & \longrightarrow & B/IB \end{array}$$

or

$$(R \to B/IB \twoheadleftarrow B) \in (R/A)_\Delta$$

Have functors $\mathcal{O}_\Delta$, $\overline{\mathcal{O}}_\Delta$ on $(R/A)_\Delta$.

$$(R \to B/IB \leftarrow B) \overset{\mathcal{O}_\Delta}{\longmapsto} B \in \{\delta\text{-alg. over } A\}$$
$$(R \to B/IB \leftarrow B) \overset{\overline{\mathcal{O}}_\Delta}{\longmapsto} B/IB \in \{R\text{-alg.}\}$$

Rmk · Even though defn of $(R/A)_\Delta$ makes sense for any R, the theory works best for R formally smooth.

Example:
(1) Say $R = A/I \Rightarrow (R/A)_\Delta = \{\text{prisms}/(A, I)\}$ has initial obj. (A, I).
(2) $R = A/I\langle x \rangle$ = p-adic. cpltion of $A/I[x]$.
$\exists\ \tilde{R}/A$ formally smooth lift of R over A/I, together w/ δ-structure.

$$\begin{array}{ccccc} A & \longrightarrow & & & \tilde{R} \\ \downarrow & & & & \downarrow \\ A/I & \longrightarrow & R & \cong & \tilde{R}/I\tilde{R} \end{array} \in (R/A)_\Delta. \quad \left(\text{e.g. } \tilde{R} = \widehat{A[x]},\ \delta(x) = 0\right)$$

(can also do same for any R).

Defn. · The prismatic coh. of R is $\Delta_{R/A} = R\Gamma((R/A)_\Delta, \mathcal{O}_\Delta) \in D(A)$. $\circlearrowleft \varphi_{R/A}$
· The Hodge-Tate coh. of R is $\overline{\Delta}_{R/A} = R\Gamma((R/A)_\Delta, \overline{\mathcal{O}}_\Delta) \in D(R)$.

These are both comm. alg. objects.

Note $\overline{\mathcal{O}}_\Delta = \mathcal{O}_\Delta \overset{L}{\underset{A}{\otimes}} A/I \rightsquigarrow \overline{\Delta}_{R/A} \cong \Delta_{R/A} \overset{L}{\underset{A}{\otimes}} A/I \in D(R)$.

ex. Say $R = A/I$, $\rightsquigarrow \Delta_{R/A} \cong A$.

The Hodge-Tate comparison.

Recall: $B \to C$, $\rightsquigarrow (\Omega^*_{C/B}, d_{dR}) = (C \xrightarrow{d} \Omega^1_{C/B} \xrightarrow{d} \Omega^2_{C/B} \xrightarrow{d} \cdots)$
= alg. dR cplx, viewed as a strictly comm. diff. graded B-alg.
$a \cdot b = (-1)^{\deg(a)\deg(b)}\, b \cdot a$
$a^2 = 0$ if $\deg(a)$ odd.

Lemma: Say (E^*, d) is a graded comm. B-~~diff~~ dga. Assume we have a map $\eta: C \to E^0$ of B-alg's.

Further, assume: $\forall x \in C$, $d(\eta(x))$ squares to 0.

$\rightsquigarrow \exists!$ extension $\eta^*: (\Omega^*_{C/B}, d) \longrightarrow (E^*, d)$.

<u>Construction</u>: Have SES of sheaves on $(R/A)_{\Delta}$:

$$0 \longrightarrow \underset{\overline{\mathcal{O}}_{\Delta}}{\overset{\shortparallel}{\mathcal{O}_{\Delta}/(d)}} \xrightarrow{(-)\cdot d} \mathcal{O}_{\Delta}/d^2 \xrightarrow{\text{can}} \underset{\overline{\mathcal{O}}_{\Delta}}{\overset{\shortparallel}{\mathcal{O}_{\Delta}/d}} \longrightarrow 0$$

$\rightsquigarrow$ get $\beta_d : H^*(\overline{\Delta}_{R/A}) \longrightarrow H^{*+1}(\overline{\Delta}_{R/A})$.

$\rightsquigarrow$ get a graded comm. (A/I)-dga $(H^*(\overline{\Delta}_{R/A}), \beta_d)$.

Have $\eta: R \longrightarrow H^0(\overline{\Delta}_{R/A})$.

Claim: $\forall x \in R$, $\beta_d(\eta(x))^2 = 0$. $\rightsquigarrow (\Omega^*_{R/(A/I)}, d_{dR}) \xrightarrow{\eta^*_R} (H^*(\overline{\Delta}_{R/A}), \beta_d)$.

Thm: (Hodge-Tate comparison) η^*_R is an isom. $\Rightarrow H^i(\overline{\Delta}_{R/A}) \simeq \Omega^i_{R/(A/I)}$ $\forall i$.

In particular, $\overline{\Delta}_{R/A} \in D_{perf}(R)$.

Rmk: (1) Gives the Cartier isom. when $I = (p)$, w/o the ~~actual~~ extra Frob. twists.

(2) can work this out explicitly for the q-dR cplx. (see notes).

How to compute coh. of cat's?

Defn. Say $\mathcal{C}$ = small cat, $PShv(\mathcal{C})$ = presheaves on $\mathcal{C}$.

Then $R\Gamma(\mathcal{C}, -)$ is the derived functor $D(Ab(\mathcal{C})) \longrightarrow D(Ab)$ of $F \longmapsto H^0 := \lim_{X \in \mathcal{C}} F(X)$.

Lemma: Assume $\exists X \in \mathcal{C}$ that is weakly-final (i.e. $Hom(Y,X) \neq \emptyset$ $\forall Y \in \mathcal{C}$). ~~Then~~ Assume $\mathcal{C}$ has finite non-empty products.

Then $R\Gamma(\mathcal{C}, F)$ is calc. by

$$F(X) \rightrightarrows F(X \times X) \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} F(X \times X \times X)$$

<u>Lemma</u>: Say (B, J) is a δ-pair over (A, I). Then $\exists$ a universal map $(B, J) \longrightarrow (C, IC)$ to a prism over (A, I).

<u>Notation</u>: $B\{\frac{J}{I}\}^\wedge := C$.

Cor: $(R/A)_{\Delta}$ has finite non-empty coproducts.

pf: Say $(R \longrightarrow B/IB \longleftarrow B)$ and $(R \longrightarrow C/IC \longleftarrow C) \in (R/A)_{\Delta}$.

Set $D_0 = B \otimes_A C$. But now have 2 maps $R \longrightarrow D_0/ID_0$.

Let $J = \ker\left(D_0 \longrightarrow B/IB \otimes_{A/IA} C/IC \longrightarrow B/IB \otimes_R C/IC\right)$

Set $D = D_0\{\frac{J}{I}\}^\wedge$. Check that (D, ID) works.

I missed 2 talks!!!

Last time: k comm. ring, $F: \mathrm{Poly}_k \longrightarrow D(Ab)$.

$\rightsquigarrow LF: \mathrm{CAlg}_k \longrightarrow D(Ab)$ "left derived functor".

$LF(A) \simeq |F(P_\cdot)|$.

$P_\cdot$: poly. resolution of A.

Example: 1) $F(A) = \Omega^1_{A/k} \rightsquigarrow LF(A) = L_{A/k}$.

2) Assume $\mathrm{char}(k) = p$. $F(A) = \Omega^*_{A/k}$.

(Recall: $H^i(\Omega^*_{A/k}) \cong \Omega^i_{A^{(1)}/k}$).

$\rightsquigarrow LF(A) =: dR_{A/k}$. derived de Rham cplx.

Prop. $\exists$ increasing filtration Fil^*_{conj} on $dR_{A/k}$, w/

$$gr_i^{conj}(dR_{A/k}) \simeq \Lambda^i L_{A^{(1)}/k}[-i].$$

Cor. If A/k is smooth, then $dR_{A/k} \simeq \Omega^*_{A/k}$.

Reason: For A smooth, have $\Lambda^i L_{A^{(1)}/k} \simeq \Omega^i_{A^{(1)}/k}$.

Derived Prismatic Cohomology

(A, I) bdd prism.

Recall: For R formally smooth A/I-alg. have $\Delta_{R/A} \circlearrowleft \varphi \in D(A)$ and isom's: $\Omega^i_{R/(A/I)} \simeq H^i(\bar{\Delta}_{R/A})$ (where $\bar{\Delta}_{R/A} = \Delta_{R/A} \overset{L}{\underset{A}{\otimes}} A/I$).

Defn. ① The derived prismatic coh.: $L\Delta_{-/A}: \mathrm{CAlg}_{A/I} \longrightarrow D(A)_{comp}$.

obtained by deriving $R \longmapsto \Delta_{\hat{R}/A}$.

$R \in \mathrm{Poly}_{A/I}$, $D(A)_{comp} = \{(p, I)\text{-complete obj. in } D(A)\}$.

Check: $L\Delta_{R/A} \simeq \Delta_{R/A}$ if R is the p-adic completion of a poly. A/I-alg.

② $L\overline{\Delta}_{R/A} := L\Delta_{R/A} \overset{L}{\underset{A}{\otimes}} A/I$. (derived Hodge-Tate coh.)

Prop. For any $R \in CAlg_{A/I}$, have an increasing exhaustive filtration Fil_*^{HT} on $L\overline{\Delta}_{R/A}$ w/ isoms:

$$gr_i^{HT}(L\overline{\Delta}_{R/A}) \simeq \Lambda^i L_{R/(A/I)}[-i]. \quad \text{in } D_{comp}(R).$$

(everything is getting completed in this picture!).

General Properties: From now on, write $\Delta_{R/A} := L\Delta_{R/A}$. (abuse of notation). (derived prismatic coh.)

1) $\Delta_{R/A} \in D_{comp}(A)$ is a comm-alg. obj. (E_∞-alg.)

2) $\overline{\Delta}_{R/A} \in D_{comp}(R)$ is a comm. alg. obj.

Perfections in mixed characteristic

Goal: Say (A,I) is a perfect prism.

(Ex: $\mathbb{Z}_p[q^{1/p^\infty}]^\wedge_{(p,q-1)}$, $([p]_q)$)

$\varphi(q) = q^p$

R is a p-complete A/I-alg.

Construct a "perfectoidization" R_{perf} of R.

(as a "derived perfectoid ring").

perfections in char. p $\quad$ k = perfect field of char. p

Defn. For any k-alg. R, $R_{perf} = \operatorname{colim}(R \xrightarrow{\varphi} R \xrightarrow{\varphi} R \to \cdots)$.

$\rightsquigarrow$ the map $R \to R_{perf}$ is universal map from R to a perfect k-alg.

Ex: $R = k[x] = \bigoplus_{i \in \mathbb{N}} k \cdot x^i \qquad R_{perf} = k[x^{1/p^\infty}] = \bigoplus_{i \in \mathbb{N}[\frac{1}{p}]} k \cdot x^i$.

Prop: For a k-alg. R, the perfection $dR_{R/k,perf} = \operatorname{colim}(dR_{R/k} \xrightarrow{\varphi} dR_{R/k} \xrightarrow{\varphi} \cdots)$ identifies w/ R_{perf} via proj $dR_{R/k} \xrightarrow{std} R$.

pf. Reduce to R = polynomial k-alg. and then use $\varphi: \Omega^i_{R/k} \to \Omega^i_{R/k}$ is 0. $\forall i > 0$.

Prop: Say $A = W(k)$, $I = (p)$, $A/I = k$. and R is a k-alg.

The perfection $\overline{\Delta}_{R/A,perf} = \operatorname{colim}(\overline{\Delta}_{R/A} \xrightarrow{\varphi} \overline{\Delta}_{R/A} \xrightarrow{\varphi} \cdots)$ identifies w/ R_{perf} via the natural map.

$$R \xrightarrow{\sim} gr_0^{HT}(\overline{\Delta}_{R/A}) \to \overline{\Delta}_{R/A}.$$

pf. use φ kills $gr_i^{HT}(\overline{\Delta}_{R/A})$ for $i > 0$

$$gr_i^{HT}(\overline{\Delta}_{R/A}) \simeq \Lambda^i L_{R/k}[-i].$$

Cor. For R, A, k as above, have a natural identification of $\Delta_{R/A,perf} = \operatorname{colim}(\Delta_{R/A} \xrightarrow{\varphi} \Delta_{R/A} \xrightarrow{\varphi} \Delta_{R/A} \xrightarrow{\varphi} \cdots)^\wedge$ and $W(R_{perf})$.

Mixed Characteristic

Notation: (A,I) = perfect prism. $D_{comp}(A) = (p,I)$-completed obj. in $D(A)$.

$D_{comp}(R)$ = p-complete obj. in $D(R)$.

Defn. (1) $\Delta_{R/A,perf} = \operatorname{colim}\left(\Delta_{R/A} \xrightarrow{\varphi} \Delta_{R/A} \xrightarrow{\varphi} \cdots\right)^\wedge \in D_{comp}(A)$.

(2) $\cancel{\Delta_{R/A}}$ $R_{perfd} = \left(\Delta_{R/A,perf}\right) \overset{L}{\underset{A}{\otimes}} A/I \overset{(?)}{\in} D_{comp}(R)$.

General properties: (1) $\Delta_{R/A,perf} \in D_{comp}(A)$ is a comm. alg. obj., and φ is an isom. on $\Delta_{R/A,perf}$.

(2) $R_{perf} \in D_{comp}(R)$ is a comm. alg. obj.

Examples: (1) This agrees w/ before if $I=(p)$.

(2). If R is already perfectoid (ex. A/I), then $R \xrightarrow{\sim} R_{perfd}$.

pf. check: $\Delta_{R/A} \xrightarrow{\sim} A_{inf}(R)$.
$$\left(\begin{tikzcd} A \arrow[r] \arrow[d] & A_{inf}(R) \arrow[d] \\ A/I \arrow[r] & R \end{tikzcd}\right).$$

(3). $(A,I) = \left(\mathbb{Z}_p[q^{1/p^\infty}]^\wedge_{(p,q-1)}, ([p]_q)\right)$.

$R = A/I[x^{\pm 1}]^\wedge$

FACT: $\Delta_{R/A} = A[x^{\pm1}]^\wedge \xrightarrow{\nabla_q} A[x^{\pm1}]^\wedge \frac{dx}{x}$

$f(x) \longmapsto \frac{f(qx)-f(x)}{q-1} \frac{dx}{x}$.

$\rightsquigarrow \Delta_{R/A,perf} = A[x^{\pm 1/p^\infty}]^\wedge \xrightarrow{\gamma-1} J \cdot A[x^{\pm 1/p^\infty}]^\wedge$.

where $J = \left(\bigcup_n (p, q^{1/p^n}-1)\right)^\wedge \subseteq A$

$= \ker\left(A \xrightarrow{q \mapsto 1} \mathbb{F}_p\right)$.

and $\gamma(x^i) = q^i x^i \quad \forall i \in \mathbb{Z}[\frac{1}{p}]$.

~~Note~~ Note: $(q-1)\cdot \underline{1} \in$ degree 1 term is a nonzero coh. class and also nonzero on R_{perfd}.

$\therefore$ R_{perfd} is genuinely derived.

General Properties: (1) $\Delta_{R/A,perf} \in D^{\geq 0}(A)$. (reason is some homotopy theory).

(2) If $\Delta_{R/A,perf} \in D^{\leq 0}(A)$ ($\therefore$ in deg 0 only).

Then $(\Delta_{R/A,perf}, I\cdot \Delta_{R/A,perf})$ is a perfect prism.

Hence R_{perfd} would be a perfectoid ring.

Example: $(A,I) = \left(\mathbb{Z}_p[q^{1/p^\infty}]^\wedge_{(p,q-1)}, ([p]_q)\right)$.

$(A,I) = \left(\mathbb{Z}_p[u^{1/p^\infty}]^\wedge_{(p,u)}, (u-p)\right)$.

Properties: • If $R_{perfd} \in D^{\leq 0}_{comp}(R)$ ($\therefore$ in deg 0) then $R \to R_{perfd}$ is the universal map to a perfectoid ring.

• R_{perfd} is independent of (A,I).

• $R \longmapsto R_{perfd}$ commutes w/ faithfully flat base changes $(A,I) \longrightarrow (B,J)$.

Thm (André): R perfectoid, $g \in R$ elt. Then $\exists$ p-completely f.f. map $R \to R_\infty$ of perfectoid rings, s.t. g admits compatible system power roots in R_∞. $\{g^{1/p^\infty}\}$.

ex. $R = \mathbb{Z}_p[p^{1/p^\infty}]^\wedge$, $g = p-1$. $(-1+x)^{\frac{1}{p}} = -1 + \cdots$

pf: $(A, I) = (A_{\inf}(R), \ker(\theta))$.

$S = \left(R[x^{1/p^\infty}] / (x-g) \right)^{\wedge}_{(p)}$.

Check: $R \to S$ is p-completely faithfully flat.

Claim: $R_\infty = S_{\mathrm{perfd}}$ solves the problem.

STS: $R \longrightarrow S_{\mathrm{perfd}}$ is (p-comp) faithfully flat.

$\Leftrightarrow$ $A \longrightarrow \Delta_{S/A, \mathrm{perf}}$ is (p, I)-comp f.f.

$\therefore$ STS: $\Delta_{S/A}$ is (p, I)-comp. f.f. over A.

as ~~statti~~ stability of flatness under $\longrightarrow$ & A is perfect.

$\Leftrightarrow$ $R \longrightarrow (\Delta_{S/A})/I$ is p-completely f.f.

Hodge-Tate comparison says $(\Delta_{S/A})/I$ has an increasing filtration w/ $gr_i \simeq \wedge^i L_{S/R}[-i]$.

$\therefore$ STS: each ~~gr~~ $\wedge^i L_{S/R}[-i]$ is p-completely f.f.

Since $\left(L_{R[x^{1/p^\infty}]/R} \right)^{\wedge} \underset{\substack{\uparrow \\ p\text{-adically}}}{\simeq} 0$. $\leftarrow$ means p-derived completion,

Have $L_{S/R} \underset{\substack{\uparrow \\ p\text{-adically}}}{\simeq} L_{S/R[x^{1/p^\infty}]} \underset{\substack{\uparrow \\ x-g \text{ is} \\ \text{a nonzerodivisor}}}{\simeq} S[1]$.

$\therefore$ get $\wedge^i L_{S/R}[-i] = \Gamma^n_S(S) \cong S$. $\square$

Cor. R perfectoid ring. $I \subseteq R$ ideal, $S = R/I$.

S_{perfd} lies in deg 0 and $S \twoheadrightarrow S_{\mathrm{perfd}}$.

ex: $R_0 = \mathbb{Z}_p^{\mathrm{cycl}} = \dfrac{\mathbb{Z}_p[q^{1/p^\infty}]}{([p]_q)}^{\wedge}$. $R = R_0[x^{1/p^\infty}]^{\wedge}$

$I = (x-1) \subseteq R$.

How to describe $\ker\left(R \longrightarrow (R/(x-1))_{\mathrm{perf}}\right)$ explicitly?

$\uparrow$ $\simeq$ almost $\mathrm{map}_{\mathrm{cts}}(\mathbb{Z}_p(1), \mathbb{Z}_p^{\mathrm{cycl}})$

$\boxed{\text{Consequently. } \begin{Bmatrix} \text{Zariski} \\ \text{closed} \end{Bmatrix} = \begin{Bmatrix} \text{strongly} \\ \text{Zariski closed} \end{Bmatrix} !!}$

The étale comparison:

(A, I) perfect prism. R p-complete A/I-alg, $d \in I$ gen.

~~(R is top. f.g. $/A/I$ and $R[p^\infty]$ is bdd)~~.

Thm. $\exists$ a canonical isom.:

$$R\Gamma_{\acute{e}t}\left(\mathrm{Spec}\left(R[\tfrac{1}{p}]\right), \mathbb{Z}/p^n\right) \xrightarrow{\sim} \left(\Delta_{R/A}[\tfrac{1}{d}]/p^n\right)^{\varphi=1}.$$

sketch of pf: ~~(S)~~ Assume $n=1$.

(1) Reduce from $\Delta_{R/A}$ to $\Delta_{R/A, \mathrm{perf}}$.

(2) Reduce to R semiperfectoid.

(3) Reduce to R perfectoid (which was known).

Step. (1): A continuity property

Notation: B is an $\mathbb{F}_p$-alg., $t \in B$ elt.

$$D(B[F]) = \{ (M, \varphi) \mid M \in D(B),\ \varphi_M : M \longrightarrow \varphi_* M \}$$

"Frobenius modules". (w/ caveat ...)

$$\cup|$$

$$D_{comp}(B[F]) = \{ (M,\varphi) \mid M \text{ is } t\text{-complete} \}.$$

$$D(B[F]) \longrightarrow D(\mathbb{F}_p).$$

$$(M,\varphi) \longmapsto M^{\varphi=1} := \mathrm{fib}(M \xrightarrow{\varphi-1} M) = R\mathrm{Hom}_{D(B[F])}((B,\varphi),(M,\varphi))$$

Note: colims in $D_{comp}(B[F])$ are computed by t-completing the usual colim.

Prop. (1) $D_{comp}(B[F]) \xrightarrow{(-)^{\varphi=1}} D(\mathbb{F}_p)$ commutes w/ colim.

(2) $(M,\varphi) \in D_{comp}(B[F])$

$$\rightsquigarrow (M,\varphi) \longrightarrow (M,\varphi)_{perf} = (\mathrm{colim}\ M \xrightarrow{\varphi} M \xrightarrow{\varphi} M \to \cdots)^\wedge$$

induces an isom on $(-)^{\varphi=1}$.

(1)': $D_{comp}(B[F]) \longrightarrow D(\mathbb{F}_p)$

$$(M,\varphi) \longmapsto (M[\tfrac{1}{t}])^{\varphi=1}$$

~~commutes w/ colim.~~

pf. say $\{(M_i,\varphi_i)\}$ is a diagram in $D_{comp}(B[F])$.

$$\begin{array}{ccc} \mathrm{colim}_i\, M_i & \xrightarrow{a} & (\mathrm{colim}\, M_i)^\wedge \\ \downarrow & & \\ \mathrm{colim}_i\, M_i[\tfrac{1}{t}] & \xrightarrow{b} & (\mathrm{colim}_i\, M_i)^\wedge[\tfrac{1}{t}]. \end{array}$$

(1) $\Leftrightarrow$ $\mathrm{fib}(a)^{\varphi=1} = 0.$

(1)' $\Leftrightarrow$ $\mathrm{fib}(b)^{\varphi=1} = 0.$

Observation: $\mathrm{fib}(a) \simeq \mathrm{fib}(b)$

(b/c $\mathrm{fib}(N \to N^\wedge)$ is uniquely t-divisible).

Hence, STS: $\mathrm{fib}(a)^{\varphi=1} = 0$ or equiv. that

$$M \longmapsto M^{\varphi=1} \text{ commutes w/ colim.}$$

Claim: For any $(N,\varphi) \in D_{comp}(B[F])$.

$N \longrightarrow N/t$ induces an isom on $(-)^{\varphi=1}$.

Note: This makes sense because $\varphi(t) = t^p \subseteq tB$.

pf. WTS: $\mathrm{fib}(N \to N/t)^{\varphi=1} = 0$.

$\mathrm{fib}(N \to N/t)$ has complete descending filtration given by "$\{t^i N\}_{i\geq 0}$" s.t. φ is top nilpotent on filtration

$\Rightarrow$ $\varphi - 1$ is an isom on $\mathrm{fib}(N \to N/t)$.

as $\varphi(t^i N)$ "$\in$" $t^{ip} N$.

Upshot: to prove étale comparison thm.

STS: $R\Gamma_{et}(\mathrm{Spec}(R[\tfrac{1}{p}]), \mathbb{F}_p) \simeq (\Delta_{R/A,perf}[\tfrac{1}{d}]/p)^{\varphi=1}$.

Step (2): Reduction to semiperfectoids

Construction: Set $T = A/I[x_1,\cdots,x_n]^\wedge$, $T_\infty := A/I[x_i^{1/p^\infty}]^\wedge$.

Have p-completely f.f. map $T \longrightarrow T_\infty$ perfectoid.

$T_\infty^* :=$ Čech nerve of $T \longrightarrow T_\infty$

$$= (T_\infty \rightrightarrows T_\infty \hat{\otimes}_T T_\infty \rightrightarrows \cdots).$$

Observation: each term is semiperfectoid (= quotient of perfectoid).

For R, choose top. generators $f_1, \cdots, f_m \in R$

$\rightsquigarrow$ get $T^{\wedge} \twoheadrightarrow R$

base change previous construction: $R \longrightarrow R_\infty^*$.

$$\begin{array}{ccc} T & \longrightarrow & T_\infty^* \\ \downarrow & & \downarrow \\ R & \longrightarrow & R_\infty^* \end{array}$$

semiperfectoids.

Strategy: reduce thm for R to thm for R_∞^*.

Lemma: $R\Gamma_{\acute{e}t}(\mathrm{Spec}(R[\frac{1}{p}]), \mathbb{F}_p) \simeq \varprojlim_{\Delta} R\Gamma_{\acute{e}t}(\mathrm{Spec}(R_\infty^*[\frac{1}{p}]), \mathbb{F}_p)$.

Lemma: $(\Delta_{R/A}[\frac{1}{d}]/p)^{\varphi=1} \simeq \varprojlim_{\Delta} (\Delta_{R_\infty^*/A}[\frac{1}{d}]/p)^{\varphi=1}$.

and also for $\Delta_{(-)/A}/p$, $\Delta_{(-)/A,\mathrm{perf}}[\frac{1}{d}]/p$.

Reduce to perfectoids

R = semi-perfectoid, R_{perfd} = perfectoidization of R

$= \Delta_{R/A,\mathrm{perf}}/d$

Lemma: R_{perfd} lives in deg 0, so is perfectoid.
(explained before)

(*): $R\Gamma_{\acute{e}t}(\mathrm{Spec}(R[\frac{1}{p}]), \mathbb{F}_p) \simeq (\Delta_{R/A}[\frac{1}{d}]/p)^{\varphi=1}$

Claim: both sides of (*) do not change if we replace R with R_{perfd}.

pf for RHS: Claim: $\Delta_{R/A,\mathrm{perf}} \simeq \Delta_{R_{\mathrm{perfd}}/A}$. use HT comp.

pf: reduce mod d to get: $R_{\mathrm{perfd}} \simeq R_{\mathrm{perfd}}$

pf for LHS: Show that $R \longrightarrow R_{\mathrm{perfd}}$ induces an isomorphism of associated arc_p-sheaves

($\rightsquigarrow$ get same étale coh. for generic fibre).

Upshot: It suffices to prove (*) for perfectoid R's;

Thm: R = perfectoid ring, then $R\Gamma_{\acute{e}t}(\mathrm{Spec}(R[\frac{1}{p}]), \mathbb{F}_p) \simeq (R^\flat[\frac{1}{d}])^{\varphi=1}$.

and $\Delta_{R/A} \simeq A_{\mathrm{inf}}(R) \rightsquigarrow \Delta_{R/A}[\frac{1}{d}]/p \simeq R^\flat[\frac{1}{d}]$ ∎

~~Thm~~

Thm $C/\mathbb{Q}_p$ complete + alg. closed, $\mathcal{O}_C \subseteq C$, $k = \mathcal{O}_C/\mathfrak{m}_C$.

$X/\mathcal{O}_C$ proper smooth formal scheme.

Then $\dim_{\mathbb{F}_p} H^i_{\acute{e}t}(X_C, \mathbb{F}_p) \leq \dim_k H^i_{dR}(X_k)$.

pf: $(A, I) = (A_{\mathrm{inf}}(\mathcal{O}_C), \ker(\theta))$. (Base).

$A_{\mathrm{inf}}(\mathcal{O}_C) \longrightarrow W(k)$. (coming from $\mathcal{O}_C \twoheadrightarrow k$).

$$\rightsquigarrow (A, I) \longrightarrow (\overset{W(k)}{\overset{\parallel}{W}}, (p)) \ldots$$

$\Delta_{X/A} \in D(X, A)$, obtained by gluing $\Delta_{R/A}$ for all affine open $\mathrm{Spf}(R) \subseteq X$.

(Note: $|X| = |X_k|$ as top. spaces)

Similarly $\Delta_{X_k/W} \in D(X_k, W)$.

Have base change: $\Delta_{X/A} \overset{\wedge L}{\underset{A}{\otimes}} W \simeq \Delta_{X_k/W}$.

(by HT comparison)

$R\Gamma_A(X) = R\Gamma(X, \Delta_{X/A}) \in D(A)$.

Claim: This is a perfect cplx.

pf: STS: $R\Gamma_A(\mathfrak{X})/d \in D_{perf}(\mathcal{O}_C)$.

But Hodge–Tate comp. $\rightsquigarrow$ $R\Gamma_A(X)/d$ is filtered w/ $gr_i \simeq R\Gamma(X, \Omega^i_{X/\mathcal{O}})[-i]$.

which is perfect as $X/\mathcal{O}$ is proper & smooth.

$V = A/p = \mathcal{O}_C^\flat$, w/ fraction field $C^\flat$, residue field k.

Semi-continuity does the job.

The q-de Rham complex

Goal: compute $\Delta_{R/A}$ on the nose. $[n]_q = \frac{q^n-1}{q-1}$.

Notation: $\mathbb{Z}_p[[q-1]] \circlearrowleft \varphi$, $\varphi(q) = q^p$, $[p]_q = \frac{q^p-1}{q-1}$, $\mathbb{Z}_p[\varepsilon_p] = \mathbb{Z}_p[[q-1]]/([p]_q)$

Construction: $R = \mathbb{Z}_p[x]^\wedge_p \rightsquigarrow$ q-dR cplx.

(Aomoto, Jackson)

$$q\Omega^*_{R,\square} = R[[q-1]] \xrightarrow{\nabla_q} R[[q-1]] \cdot dx$$

$$\nabla_q(f(x)) = \frac{f(qx) - f(x)}{qx - x}\, dx \quad (\text{Note } f(qx) \equiv f(x) \bmod (qx - x)).$$

$$\nabla_q(x^n) = \frac{q^n x^n - x^n}{qx - x} = [n]_q \cdot x^{n-1} \cdot dx.$$

$$\rightsquigarrow q\Omega^*_{R,\square}/(q-1) \xrightarrow{\sim} \Omega^*_{R/\mathbb{Z}_p}.$$

on the nose.

Rmk: check $q\Omega^*_{R,\square}$ is not q-isom. to $\Omega^*_{R/\mathbb{Z}_p} \hat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[[q-1]]$.

Rmk (multiplicative structure): ∇_q satisfies the q-Leibniz rule:

$$\nabla_q(f(x)g(x)) = f(x) \cdot \nabla_q(g(x)) + g(qx) \cdot \nabla_q(f(x)).$$

$\rightsquigarrow$ $q\Omega^*_{R,\square}$ is a dga if we make $q\Omega^1_{R,\square}$ into a $q\Omega^0_{R,\square}$-bimodule by

non-commutative!

$$a(x) \underset{L}{\cdot} \omega \underset{R}{\cdot} b(x) = a(x) \cdot b(qx) \cdot \omega$$

It will turn out to be commutative up to all possible homotopies.

formally étale

Defn: A *framing* is a formally étale map

$$\mathbb{Z}_p[X_1, \cdots, X_n]^\wedge \longrightarrow S$$

of formally smooth $\mathbb{Z}_p$-algebras.

Call $(S, \square)$ a framed pair.

Construction (q-dR cplx for framed pairs)

from $(S, \square)$, get :

$$\tilde{\square} : \mathbb{Z}_p[q-1, X_1, \cdots, X_n]^\wedge_{(p, q-1)} \longrightarrow S[[q-1]].$$

formally étale map of $\mathbb{Z}_p[[q-1]]$-alg.

For $i \in \{1, \cdots, n\}$, have an aut. γ_i of

$$\mathbb{Z}_p[q-1, X_1, \cdots, X_n]^\wedge_{(p, q-1)}$$

$$\gamma_i(X_j) = \begin{cases} X_j & j \neq i \\ q \cdot X_i & j = i. \end{cases}$$

as $\gamma_i \equiv id \mod (qX_i - X_i)$, get a unique aut. γ_i of $S[[q-1]]$ extending it.

Now define $\nabla_{q,i}(f \in S) = \dfrac{\gamma_i(f) - f}{qX_i - X_i} dX_i$.

$$\rightsquigarrow q\Omega^*_{S,\square} = \text{Koszul cplx}(S[[q-1]], \nabla_{q,1}, \nabla_{q,2}, \cdots, \nabla_{q,n})$$

q-de Rham cplxes.

Recall: $S/\mathbb{Z}_p$ formally smooth.

$\square : \mathbb{Z}_p[x_i]^\wedge \longrightarrow S$. formally étale.

$q\Omega^*_{S,\square} = \mathrm{Kos}(S[[q-1]], \nabla_{q,1}, \dots, \nabla_{q,n})$.

$\nabla_{q,i}(f) = \dfrac{\gamma_i(f) - f}{qx_i - x_i}$, where $\gamma_i(x_j) = \begin{cases} x_j & j \neq i \\ qx_i & j = i. \end{cases}$

Lemma: $(q\Omega^*_{S,\square})/(q-1) \cong \Omega^*_{S/\mathbb{Z}_p}$. i.e. $\nabla_{q,i}(f) = \dfrac{df}{dx_i} \bmod (q-1)$.

Conj (Scholze). $\exists$ a symmetric monoidal functor $S \mapsto q\Omega_S$ from formally smooth $\mathbb{Z}_p$-alg. to $D_{comp}(\mathbb{Z}_p[[q-1]])$.
equipped w/ natural isoms — complete w.r.t. $(p, q-1)$.

$q\Omega_S \simeq q\Omega^*_{S,\square}$ for each choice of framing.

In particular, $q\Omega_S$ is a comm. alg. in $D_{comp}(\mathbb{Z}_p[[q-1]])$.

$\rightsquigarrow$ $q\Omega^*_{S,\square}$ is an E_∞-alg.

Rmk: 1) Conj. is closely related to integral p-adic Hodge theory, which proves conjecture after base change along

$$\mathbb{Z}_p[[q-1]] \longrightarrow \mathbb{Z}_p[q^{1/p^\infty}]^\wedge_{(p,q-1)}.$$

2) Conj. is "easy" after base change along $\mathbb{Z}_p[[q-1]] \to \mathbb{Q}_p[[q-1]]$.

3) Conj makes sense if we replace $\mathbb{Z}_p$ w/ any (p-cplt) ring, but is not true in this generality: it fails for $\mathbb{F}_p$.

q-Crystalline cohomology

Goal: Construct a q-Crystalline site $(R/\mathbb{Z}_p[[q-1]])_{q\text{Crys}}$ whose cohomology is computed by q-dR cplxes.

Notation: $A = \mathbb{Z}_p[[q-1]] \circlearrowleft \varphi : \varphi(q) = q^p$.
$A \longrightarrow \mathbb{Z}_p$, $q \longmapsto 1$.
$\mathbb{Z}_p[\varepsilon_p] = A/([p]_q)$

obs: $(p, q-1)$ and $(p, [q]_p)$ define the same topology.

$\boxed{q\text{-PD thickenings}}$

Defn: A q-PD pair (D, I) is a pair where D is a δ-A-alg, $I \subseteq D$ is an ideal s.t.
a) D and D/I are cplt w.r.t. $(p, [p]_q)$.
b) D is $[p]_q$-torsionfree.
c) I contains $q-1$ and $\varphi(I) \subseteq [p]_q D$.

Call $D \to D/I$ the corresponding q-PD thickening.

example: (1) $(A, (q-1))$ is a q-PD pair (and is the initial such pair).
More generally, for any $(p, [p]_q)$-completely flat δ-A-alg D, get a q-PD pair $(D, (q-1))$.

(2) $(q=1)$: if $q=1$ in D, then
$(c) \Leftrightarrow \forall x \in I, \ \frac{x^n}{n!} \in D \ \forall n \geq 1$.
lemma on divided powers.

Lemma: Say D is $[p]_q$-torsionfree δ-A-alg. Let $f \in D$, s.t.
$\varphi(f) \in [p]_q D \rightsquigarrow \varphi\left(\frac{\varphi(f)}{[p]_q} - \delta(f)\right) \in [p]_q D$.

Note: If $q=1$, $\frac{\varphi(f)}{[p]_q} - \delta(f) = \frac{f^p + p\delta(f)}{p} - \delta(f) = \frac{f^p}{p}$.
Lemma $\Rightarrow \left(f^p \in pD \underset{(?)}{\rightsquigarrow} f^{p^2} \in p^{p+1} D\right)$.

pf: reduce to the universal case, where D is A-flat.
Goal: $\frac{\varphi^2(f)}{\varphi([p]_q)} \equiv \varphi(\delta(f)) \mod [p]_q D$.

In $D/[p]_q D$, the elt $\varphi([p]_q) = p$ is a non-zerodivisor.
$\varphi([p]_q) = \frac{q^{p^2}-1}{q^p-1}$

STS: $\varphi^2(f) \equiv p\,\varphi(\delta(f)) \mod [p]_q D$.
Have $\varphi(f) = f^p + p\delta(f)$
Apply φ: $\varphi^2(f) = \varphi(f)^p + p\,\varphi(\delta(f))$.
$\varphi(f)^p \in [p]_q D$ by assumption.

Prop. (Existence of q-PD envelopes) Say R is a formally smooth $\mathbb{Z}_p$-alg. Say P is a formally smooth δ-A-alg, equipped w/ a surj. $P \twoheadrightarrow R$ with kernel J.
Then $\exists$ a universal map $(P, J) \longrightarrow (D, I)$ to a q-PD pair. It has the following features:
(a) D is $(p, [p]_q)$-completely flat $/A$.
(b) The map $(P, J) \longrightarrow (D, I)$ gives an isom $R = P/J \xrightarrow{\sim} D/I$.
(c) The map $D/(q-1) \longrightarrow R$ is the (p-completed) PD-env. of $P/(q-1) \twoheadrightarrow R$.

Write $D_{J,q}(P) = D$.

pf: $D = P\left\{\frac{\varphi(x_1)}{[p]_q}, \dots, \frac{\varphi(x_n)}{[p]_q}\right\}^\wedge_{(p,[p]_q)}$. where $J = (q-1, \underbrace{x_1, \dots, x_n}_{\text{reg. seq.}})$

ex: Say $R = \mathbb{Z}_p[t]^\wedge$, $P = A[x,y]^\wedge$, $P \to R$, $x, y \mapsto t$. $\delta(x) = \delta(y) = 0$.

$J = (q-1, x-y)$.

fact: $D_{J,q}(P)$ contains $\gamma_{k,q}(x-y) = \frac{(x-y)(x-qy)\cdots(x-q^{k-1}y)}{[k]_q[k-1]_q\cdots[1]_q}$ and form a topological basis over $A[x]^\wedge$.

$\boxed{q\text{-Crystalline Site}}$

Defn. Say R is formally smooth $\mathbb{Z}_p$-alg.

$$(R/A)_{q\mathrm{Crys}} = \left\{ (D, I) \ \ q\text{-PD pair and } D/I \simeq R \right\}.$$

Make it into a site via indiscrete topology (so Presheaves = Sheaves).

$(D, I) \mapsto D$ gives a sheaf $\mathcal{O}_{q\mathrm{Crys}}$.

$$q\Omega_R = R\Gamma((R/A)_{q\mathrm{crys}}, \mathcal{O}_{q\mathrm{Crys}}) \in D_{\mathrm{comp}}(A).$$

Construction: (Cech-Alexander complexes for $q\Omega_R$) Choose a surj. $P \twoheadrightarrow R$ w/ kernel J s.t. P is a $(p, [p]_q)$-~~completely~~ cpltion of a free δ-A-alg.

Get a cosimplicial δ-A-alg.

$$P^\bullet = (P \rightrightarrows P \hat{\otimes}_A P \cdots) \text{ with an ideal } J^\bullet \subseteq P^\bullet$$

s.t. $P^\bullet/J^\bullet = R$.

Prop $\Rightarrow$ $D_{J^\bullet,q}(P^\bullet) = \left(D_{J^0,q}(P^0) \rightrightarrows D_{J^1,q}(P^1) \cdots \right)$

cosimplicial obj of $(R/A)_{q\mathrm{crys}}$.

Category theory $\Rightarrow$ $q\Omega_R$ is computed by $D_{J^\bullet;q}(P^\bullet)$.

Thm ($q=1$ specialization) $\exists$ a canonical isom. : $q\Omega_R \overset{L}{\underset{A}{\otimes}} \mathbb{Z}_p \simeq \Omega^*_{R/\mathbb{Z}_p}$.

pf: $\cdot$ $\Omega^*_{R/\mathbb{Z}_p} \simeq R\Gamma_{\mathrm{crys}}((R/p)/\mathbb{Z}_p)$.

$\cdot$ $R\Gamma_{\mathrm{crys}}((R/p)/\mathbb{Z}_p)$ is ~~compl~~ computed by $D_{J^\bullet;q}(P^\bullet)/(q-1)$ (by Cech-Alexander for crys coh. + Prop. (c).)

which also computes LHS.

Last time: $A = \mathbb{Z}_p[[q-1]] \longrightarrow \mathbb{Z}_p$, $q \longmapsto 1$, $A \circlearrowleft \varphi$

R formally smooth$/\mathbb{Z}_p$

$$(R/A)_{q\text{-crys}} = \left\{ \begin{tikzcd} A \arrow[r] \arrow[d] & D \arrow[d] \arrow[loop right, "\varphi"] \\ \mathbb{Z}_p \arrow[r] & R \end{tikzcd} \;\middle|\; \cancel{\exists}\ \varphi(\ker(D\to R)) \subseteq [p]_q \cdot D \right\}.$$

$$q\Omega_R = R\Gamma\left((R/A)_{q\text{crys}}, \mathcal{O}\right).$$

Thm: $q\Omega_R/(q-1) \simeq \Omega^*_{R/\mathbb{Z}_p}$

Key tool: Given $P \twoheadrightarrow R$, P formally smooth $/A$ w/ δ-structure.
$\exists$ universal map $P \longrightarrow D_{J,n}(P)$

$$\begin{tikzcd} P \arrow[r] \arrow[dr] & D_{J,n}(P) \arrow[d] \\ & R \end{tikzcd} \leftarrow q\text{-PD thickening.}$$

Thm. (Prismatic comparison) R formally sm. $/\mathbb{Z}_p$. Set $R^{(1)} := R \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\zeta_p] \leftarrow A/[p]_q$.

$$\left(\begin{tikzcd} A \arrow[r, "\varphi_A"] \arrow[d] & A \arrow[d] & \\ \mathbb{Z}_p \arrow[r] \arrow[d] & A/_{q^p-1} \arrow[r] & A/[p]_q \arrow[d] \\ R \arrow[rr] & & R^{(1)} \end{tikzcd}\right)$$

Then $\exists$ a natural ~~canonical~~ isom

$$q\Omega_R \simeq \Delta_{R^{(1)}/A}.$$

pf sketch: to get a map $\Delta_{R^{(1)}/A} \longrightarrow q\Omega_R$, need to show

each $(D\to R) \in (R/A)_{q\text{crys}}$

get an obj. $(R^{(1)} \longrightarrow D/[p]_q \longleftarrow D) \in (R^{(1)}/A)_{\Delta}$.

Have
$$\begin{tikzcd} D\ (\ker = J) \arrow[r, "\varphi_D"] \arrow[d] & D \arrow[d] \\ R \arrow[r] & D/[p]_q D \end{tikzcd} \quad \varphi_D(J) \subseteq [p]_q D.$$

Frobenius linear $\leftarrow$ Linearizing get $R^{(1)} \longrightarrow D/[p]_q D$.

after getting $\Delta_{R^{(1)}/A} \longrightarrow q\Omega_R$.
To check it's an isom., check after $-\otimes^L_A A/(q-1)$.
(and using crystalline-prismatic comparison). $\square$

Goal: relate $q\Omega_R$ to $q\Omega^*_{R,\square} \leftarrow$ q-de Rham cplxes

Construction: (q-de Rham comparison for q-PD envelopes) Say P is a formally smooth A-alg. formally étale $/A[x_1,\dots,x_n]$ $\rightsquigarrow$ q-dR complexes $q\Omega^*_{P,\square}$.
unique δ-structure on P via $\delta(x_i)=0$

Assume $\exists$ a surj. $P \twoheadrightarrow R \leftarrow$ smooth$/\mathbb{Z}_p$

$D = q$-PD-envelope of $P \longrightarrow R$.

Claim: each q-derivative $\nabla_{q,i}: P \longrightarrow P$ extends uniquely to D.

pf: $\nabla_{q,i}(f) = \dfrac{\gamma_i(f) - f}{qx_i - x_i}$ where $\gamma_i: P \longrightarrow P$, $x_i \longmapsto qx_i$, $x_j \longmapsto x_j$.

STS: each γ_i extends uniquely to D which is congruent to identity mod $(qx_i - x_i)$.
using the univ. property of D, enough to show

(*) $\forall f \in J = \ker(P \to R)$, have

- $\varphi(\gamma_i(f)) \subseteq [p]_q D$ and
- $\dfrac{\varphi(\gamma_i(f))}{[p]_q} = \dfrac{\varphi(f)}{[p]_q}$ mod $(qx_i - x_i)\cdot D$.

Know: $\gamma_i(f) = f + (qx_i - x_i)g \in P$.

$\varphi(\gamma_i(f)) = \varphi(f) + (q^p - 1)x_i^p\varphi(g) \in P$.

$\Rightarrow \dfrac{\varphi(\gamma_i(f))}{[p]_q} = \dfrac{\varphi(f)}{[p]_q} + (q-1)\cdot x_i^p \cdot \varphi(g)$.

Upshot: can make sense of $q\Omega^*_{D,\square} = \mathrm{Kos}(D; \{\gamma_{q,i}\}_{i=1,\dots,n})$.

Thm: $(R,\square)$ framed $\mathbb{Z}_p$-alg ($\square : \mathbb{Z}_p[\underline{x}] \to R$, formally étale). Then, $\exists$ natural isom:

$$q\Omega_R \simeq q\Omega^*_{R,\square}.$$

pf sketch: Let P = unique lift of R to A w/ coordinates $x_1, \dots, x_n$.

$\rightsquigarrow P^\bullet = \mathrm{Cech}(A \to P)$.
$= (P \rightrightarrows P \hat{\otimes}_A P \rightrightarrows \cdots)$.

Have surjection $P^n \to P \to R$.
with kernel J^n.

$\rightsquigarrow D_{J^\bullet,q}(P^\bullet)$ - cosimplicial δ A-alg.
computing $q\Omega_R$.

Draw a bicomplex $q\Omega^*_{D_{J^\bullet,q}}(P^\bullet)$:

$$\begin{tikzcd}
 & \vdots & \vdots & \\
\Omega^1_{P^0} = & \Omega^1_{P^0}\otimes_{P^0} D_{J^0,q}(P^0) \arrow[r, shift left] \arrow[r, shift right] \arrow[u] & \Omega^1_{P^1}\otimes_{P^1} D_{J^1,q}(P^1) \arrow[r, shift left] \arrow[r] \arrow[r, shift right] \arrow[u] & \cdots \\
P^0 = \arrow[u] & D_{J^0,q}(P^0) \arrow[r, shift left] \arrow[r, shift right] \arrow[u, "\nabla_q"] & D_{J^1,q}(P^1) \arrow[r, shift left] \arrow[r] \arrow[r, shift right] \arrow[u] & \cdots
\end{tikzcd}$$

So we must show 1st column $\underset{qis}{\simeq}$ 1st row.

This follows by combining:

a) all horizontal maps gives quasi-isom's of the columns.

~~This~~ pf: reduce $\bmod\ (q-1)$, and use Poincaré Lemma.

$\rightsquigarrow$ bicomplex totalize to $q\Omega^*_{R,\square}$ (which is the 1st column).

b) all rows except 0th row are acyclic.

$\rightsquigarrow$ totalize to 1st row: $q\Omega_R$.

get $q\Omega_R \simeq q\Omega^*_{R,\square}$.

(It would be fun to chase diagram, just a little bit).

Rmk: may use q-dR complexes to show that

$\varphi_A^* \Delta_{R/A} \xrightarrow{\varphi_{R/A}} \Delta_{R/A}$ is an isogeny.

(i.e. has an inverse up to multiplication by I^d, $d = \dim(R/A)$)

(A, I) prism.